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CERTAIN RESULTS INVOLVING CONTINUED FRACTIONS AND MODULAR IDENTITIES

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Abstract: In this paper certain results on continued fractions and modular identities have been established.

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1. Introduction

Let a and q be variables such that $|q| < 1$, then the conventional $q-$ Pochammer symbol is defined as,

$$(a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k)$$

for any positive integer n and

$$(a;q)_0 = 1 \quad \text{and} \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Also, for any real number n we define,

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$$

and for variables $a_1, a_2, ..., a_k$ we define the shorthand notation

$$(a_1, a_2, ..., a_k; q)_n = (a_1;q)_n(a_2;q)_n...(a_k;q)_n.$$

The study of finite continued fractions i.e., expressions of the form

$$\cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}}$$

which is written more economically as,

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n}$$

began in its explicit form in the letter decades of the 16th century with a paper of Bombelli written when the concepts and notations of algebra were first being laid down in Italy and France.

Thus use of continued fractions as an important tool in number theory began with the 17th century results of Schwenter, Huygens and Walls and come to maturity with the work of Euler in 1737. The infinite continued fractions is written as,

$$\frac{a_1}{b_1+}\frac{a_2}{b_2+}\frac{a_3}{b_3+}\cdots\frac{a_n}{b_n+}\cdots\infty$$

One of the old result on continued fraction is,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\cdots\infty \tag{1.1}$$

Proof of this result is very simple, viz.,

$$\frac{\sqrt{5}-1}{2} = \frac{1}{\dfrac{2}{\sqrt{5}-1}\left(\dfrac{\sqrt{5}+1}{\sqrt{5}+1}\right)}$$

$$= \frac{1}{\dfrac{\sqrt{5}+1}{2}} = \frac{1}{1+\dfrac{\sqrt{5}-1}{2}}$$

Now, iterating this process one can get (1.1).
This result might has attracted Ramanujan. He has established large number of results involving continued fraction in his first, second, third and also in 'lost' Notebooks. Many other mathematicians all over the world have either established or proved the results of Srinivasa Ramanujan. Some of the notable names of the mathematicians are as Andrews [2], Bruce Berndt [7, 8], Agarwal R. P. [1], Adiga, C., Berndt, B. C., Bhargava and Watson, G. N. [3], Andrwes G. E. and Bowmen D., [4], Bhagirathi [9], Bhargava, S., Adiga C., and Somashekare, D. [10], Denis, R. Y. [11, 12], Denis, R. Y. and Singh, S. N. [13], Denis R. Y. and Singh, S. P. [14], Singh, S. N. [17, 18] and many others.

Now following the process of (1.1) we establish a new continued fraction of $\dfrac{H_1(q)}{G_1(q)}$.

2. Continued Fraction

In this section we establish a result on continued fraction. The most celebrated identities due to Rogers and Ramanujan are

$$H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}, \tag{2.1}$$

and

$$G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}. \tag{2.2}$$

Ramanujan showed that

$$\frac{H(q)}{G(q)} = \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n}} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}$$

$$= \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\dots_\infty, \qquad \text{where} \quad |q|<1. \tag{2.3}$$

Ramanujan established following two more identities,

$$H_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(q^2;q^2)_\infty}{(q^2,q^3;q^5)_\infty}, \tag{2.4}$$

and

$$G_1(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} = \frac{(q^2;q^2)_\infty}{(q,q^4;q^5)_\infty}. \tag{2.5}$$

Taking the ratio of $H_1(q)$ and $G_1(q)$ we have,

$$\frac{H_1(q)}{G_1(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty}. \tag{2.6}$$

Now, making use of (2.3) we have

$$\frac{H_1(q)}{G_1(q)} = \frac{(q,q^4;q^5)_\infty}{(q^2,q^3;q^5)_\infty} = \frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\dots_\infty, \qquad \text{where} \quad |q|<1. \tag{2.7}$$

Now following the process of (1.1) we establish a new continued fraction of $\frac{H_1(q)}{G_1(q)}$.
Let us consider

$$\frac{H_1(q)}{G_1(q)} = \frac{\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}}$$

$$= \frac{1}{1 + \frac{\sum_{n=0}^{\infty} \frac{q^{n^2}(1-q^{2n})}{(q^2;q^2)_n(-q^2;q^2)_n}}{\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^2;q^2)_n}}}$$

$$= \cfrac{1}{1 + \cfrac{\displaystyle\sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^2;q^2)_{n-1}(-q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^2;q^2)_n}}}$$

$$= \cfrac{1}{1 + \cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^2;q^2)_n(-q^2;q^2)_{n+1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^2;q^2)_n}}}$$

$$= \cfrac{1}{1 + \cfrac{q/(1+q^2)}{1 + \cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n} \left\{ \frac{1}{(-q^2;q^2)_n} - \frac{1}{(-q^4;q^2)_n} \right\}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^4;q^2)_n}}}}$$

$$= \cfrac{1}{1 + \cfrac{q/(1+q^2)}{1 - \cfrac{\displaystyle\sum_{n=1}^{\infty} \frac{q^{n^2+2n+2}}{(q^2;q^2)_{n-1}(-q^2;q^2)_{n+1}}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(-q^4;q^2)_n}}}}$$

$$= \cfrac{1}{1 + \cfrac{q/(1+q^2)}{1 - \cfrac{q^5/(1+q^2)(1+q^4)}{1 + \cfrac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+2n}(1-q^{2n})}{(q^2;q^2)_n(-q^4;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n^2+4n}}{(q^2;q^2)_n(-q^4;q^2)_n}}}}}$$

$$= \cfrac{1}{1+\cfrac{q/(1+q^2)}{1-\cfrac{q^5/(1+q^2)(1+q^4)}{1+\cfrac{q^3/(1+q^4)}{1+\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+4n}}{(q^2;q^2)_n}\left\{\frac{1}{(-q^4;q^2)_n}-\frac{1}{(-q^6;q^2)_n}\right\}}{\sum_{n=0}^{\infty}\frac{q^{n^2+4n}}{(q^2;q^2)_n(-q^6;q^2)_n}}}}}}$$

$$= \cfrac{1}{1+\cfrac{q/(1+q^2)}{1-\cfrac{q^5/(1+q^2)(1+q^4)}{1+\cfrac{q^3/(1+q^4)}{1-\cfrac{q^7/(1+q^4)(1+q^6)}{1+\cfrac{\sum_{n=0}^{\infty}\frac{q^{n^2+4n}}{(q^2;q^2)_n(-q^6;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{n^2+6n}}{(q^2;q^2)_n(-q^6;q^2)_n}}}}}}}$$

Iterating this process we get,

$$\frac{H_1(q)}{G_1(q)} = \cfrac{1}{1+\cfrac{q/(1+q^2)}{1-\cfrac{q^5/(1+q^2)(1+q^4)}{1+\cfrac{q^3/(1+q^4)}{1-\cfrac{q^7/(1+q^4)(1+q^6)}{1+\cfrac{q^5/(1+q^6)}{1-\cfrac{q^9/(1+q^6)(1+q^8)}{1+\ddots}}}}}}}$$

$$\frac{H_1(q)}{G_1(q)} = \frac{1}{1+}\,\frac{q/(1+q^2)}{1-}\,\frac{q^5/(1+q^2)(1+q^4)}{1+}\,\frac{q^3/(1+q^4)}{1-}\,\frac{q^7/(1+q^4)(1+q^6)}{1+}\,\frac{q^5/(1+q^6)}{1-}\,\frac{q^9/(1+q^6)(1+q^8)}{1+}\dots \quad . \tag{2.8}$$

3. Modular Identities and Continued Fractions

In this section we discuss about certain results involving modular identities and continued fractions.
Here we start with Ramanujan's theta functions. Ramanujan motivated with Jacobi's theta functions defined his own theta functions as,

$$f(a,b)=\sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab|<1. \tag{3.1}$$

Applying the Jacobi's triple product identity [15; App. II 28] (3.1) yields following functions,

$$\begin{aligned} f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \\ &= \sum_{n=-\infty}^{\infty} (ab)^{n^2/2}\left(\frac{a}{b}\right)^{n/2} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \end{aligned} \tag{3.2}$$

Following are the particular cases of (3.2),

$$f(q,q)=\Phi(q)=(-q;q^2)_\infty^2(q^2;q^2)_\infty. \tag{3.3}$$

$$f(q,q^3)=\Psi(q)=(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty=(-q;q)_\infty(q^2;q^2)_\infty. \tag{3.4}$$

$$f(-q)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_\infty. \tag{3.5}$$

In order to establish our main results we need following modular identities

$$G(q)H(-q)+G(-q)H(q)=\frac{2\Psi(q^2)}{(q^2;q^2)_\infty}. \tag{3.6}$$

[5; (4.3.5) p. 114]

$$G(q)H(-q)-G(-q)H(q)=\frac{2q\Psi(q^{10})}{(q^2;q^2)_\infty}. \tag{3.7}$$

[5; (4.3.6) p. 114]

In the 'Lost' notebook of Ramanujan mentioned following identity,

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\Phi(q)}{(q^2;q^2)_\infty}. \tag{3.8}$$

By making use of (3.3)

$$G(q)G(q^4) + qH(q)H(q^4) = (-q;q^2)_\infty. \tag{3.9}$$

G. N. Watson proved (3.8) and also established following two identities.

$$G(-q)\Phi(q) - G(q)\Phi(-q) = 2qH(q^4)\Psi(q^2). \tag{3.10}$$

$$H(-q)\Phi(q) + H(q)\Phi(-q) = 2G(q^4)\Psi(q^2). \tag{3.11}$$

(a) Adding (3.6) and (3.7) we get,

$$G(q)H(-q) = \frac{\Psi(q^2) + q\Psi(q^{10})}{(q^2;q^2)_\infty}. \tag{3.12}$$

Again, subtracting (3.7) from (3.6) we get,

$$G(-q)H(q) = \frac{\Psi(q^2) - q\Psi(q^{10})}{(q^2;q^2)_\infty}. \tag{3.13}$$

Dividing (3.13) by (3.12) we find,

$$\frac{\Psi(q^2) - q\Psi(q^{10})}{\Psi(q^2) + q\Psi(q^{10})} = \frac{H(q)/G(q)}{H(-q)/G(-q)} \tag{3.14}$$

Now making use of (2.3) in (3.14) we have,

$$\frac{\Psi(q^2) - q\Psi(q^{10})}{\Psi(q^2) + q\Psi(q^{10})} = \left\{\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\cdots\right\}\left\{\frac{1}{1-}\frac{q}{1+}\frac{q^2}{1-}\frac{q^3}{1+}\cdots\right\}^{-1}. \tag{3.15}$$

Putting $-q$ for q in (3.8) we have,

$$G(-q)G(q^4) - qH(-q)H(q^4) = \frac{\Phi(-q)}{(q^2;q^2)_\infty} = (q;q^2)^2_\infty. \tag{3.16}$$

Dividing (3.9) by (3.16) we get

$$\frac{G(q)G(q^4) + qH(q)H(q^4)}{G(-q)G(q^4) - qH(-q)H(q^4)} = \frac{(-q;q^2)^2_\infty}{(q;q^2)^2_\infty}$$

$$= (-q;q^2)^2_\infty(-q;q)^2_\infty = \{(-q;q)_\infty(-q;q^2)_\infty\}^2. \tag{3.17}$$

Dividing (3.10) by (3.11) we find,

$$\frac{G(-q)\Phi(q) - G(q)\Phi(-q)}{H(-q)\Phi(q) + H(q)\Phi(-q)} = q\frac{H(q^4)}{G(q^4)}$$

$$= \frac{q}{1+}\frac{q^4}{1+}\frac{q^8}{1+}\frac{q^{12}}{1+}\frac{q^{16}}{1+}..., \tag{3.18}$$

which is known result [16; (2.10), p. 208].
Again, multiplying (3.10) by $H(q)$ and (3.11) by $G(q)$ and adding them we find,

$$\Phi(q)\left\{H(q)G(-q) + G(q)H(-q)\right\} = 2\Psi(q^2)\left\{H(q)H(q^4) + G(q)G(q^4)\right\}. \tag{3.19}$$

Now, using Ramanujan's modular identity (3.8) in (3.19) we get,

$$H(q)G(-q) + G(q)H(-q) = \frac{2\Psi(q^2)}{(q^2;q^2)_\infty} = 2(-q^2;q^2)^2_\infty, \tag{3.20}$$

which is also a known result [16; (2.6), p. 207].

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